

## ASYMPTOTIC BEHAVIOR OF THE NUMBER OF LOST MESSAGES\*

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**Abstract.** The goal of the paper is to study asymptotic behavior of the number of lost messages. Long messages are assumed to be divided into a random number of packets which are transmitted independently of one another. An error in transmission of a packet results in the loss of the entire message. Messages arrive to the  $M/GI/1$  finite buffer model and can be lost in two cases as either at least one of its packets is corrupted or the buffer is overflowed. With the parameters of the system typical for models of information transmission in real networks, we obtain theorems on asymptotic behavior of the number of lost messages. We also study how the loss probability changes if redundant packets are added. Our asymptotic analysis approach is based on Tauberian theorems with remainder.

**Key words.** loss systems,  $M/GI/1/n$  queue, busy period, redundancy, loss probability, asymptotic analysis, Tauberian theorems with remainder

**AMS subject classifications.** 60K25, 60K30, 40E05

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### 1. Introduction.

#### 1.1. Review of the literature and general description of the system.

Long messages in Internet protocols that have to be transmitted are divided into small packets. Upon transmission each packet is transformed by providing additional information related to a given message. Because of the bit errors in transmission of the packet, the message can be lost. The loss probability of a message plays a significant role in the evaluation of network performance and design of network topology.

There are a number of papers in which the loss probability of a message has been studied. Cidon, Khamisy, and Sidi [11] derived recurrence relations for the loss probabilities of packets in a message giving the numerical results for the  $M/M/1/n$  buffer model. The complexity of recurrence calculations of that paper are  $O(nm^2)$ , where  $m$  is the size of a message and  $n$  is the buffer capacity. Considering the same model, Gurewitz, Sidi, and Cidon [13] obtained another representation for the loss probability by using the ballot theorem (e.g., Takács [17]). In the framework of the same model Altman and Jean-Marie [7] give a comprehensive analysis for the multidimensional generating function of the loss probabilities based on the recurrence relations of the paper of Cidon, Khamisy, and Sidi [11] and analyze the effect of adding redundant packets. Studying a slightly more general model with several sources, Ait-Hellal et al. [6] obtained some asymptotic results and studied the effect of adding redundancy to the loss probability. The aforementioned papers [6], [7], [11], [13] all discuss the problem of complexity of calculations as well as the required memory to store intermediate variables.

In real communication networks the capacity is large. Therefore, asymptotic analysis of the number of lost messages is necessary. The present paper provides asymptotic analysis with sequential application to redundancy of the following model. Assume that each message is divided into a random number of packets each of which

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is forwarded to the buffer. For the  $i$ th message denote its random number of packets by  $\nu_i$ . We assume that the sequence  $\nu_i \geq 1, i \geq 1$ , consists of independently and identically distributed integer random variables. The interarrival times between messages have an exponential distribution with parameter  $\lambda$ . The buffer can contain only  $N$  packets; that is, if immediately before the arrival of message of  $l$  packets there are  $L$  packets in the buffer, then the message is accepted only if  $L + l \leq N$ ; otherwise the message of  $l$  packets is lost. The loss of a message can also occur if at least one packet in a message is corrupted. In this case we assume that if there is enough space, then the message does occupy the buffer, but it is hidden and therefore lost. The probability that at least one packet in a message is corrupted is denoted by  $p$ .

In general loss communication networks, a transmission time typically depends on the number of packets in a message. To be realistic we must study a general queueing system with service time depending on batch size. The analysis of such a system is a hard problem. On the other hand, the model with a fixed number of packets in a message, leading to the standard  $M/GI/1/n$  queueing system, is not realistic. Therefore, in the following we assume additionally that the random variables  $\nu_i$  have fixed upper and lower bounds  $\nu^{upper}$  and  $\nu^{lower}$ , i.e.,  $\mathbf{P}\{\nu^{lower} \leq \nu_i \leq \nu^{upper}\} = 1$ . This assumption can be considered as a compromise between these two cases. It has a real application in some communication technologies, especially in optical local networks, where a number of small messages following the same direction are combined as one message (bus).<sup>1</sup> Outgoing from the local network, the bus continues on its way being processed by the Internet protocols. When the difference between  $\nu^{upper}$  and  $\nu^{lower}$  for the messages is not large, then assuming that a transmission time is independent of the message size seems appropriate.

**1.2. Formulation of the model in terms of the queueing theory.** In terms of the queueing theory the model can be described as follows. We assume that messages arrive to the finite buffer  $M/GI/1$  queue with random number of waiting places  $\zeta$ . The input rate is equal to  $\lambda$ , and the service time distribution is  $B(x)$  with the expectation  $b$ . By a queueing system with random number of waiting places we mean the following. We denote

$$\zeta = \inf \left\{ m : \sum_{i=1}^m \nu_i \leq N \right\},$$

and according to the assumption  $\mathbf{P}\{\nu^{lower} \leq \nu_i \leq \nu^{upper}\} = 1$ , there are two fixed values  $\zeta^{upper}$  and  $\zeta^{lower}$  depending on  $N$ , and  $\mathbf{P}\{\zeta^{lower} \leq \zeta \leq \zeta^{upper}\} = 1$ .

Let  $\zeta_1, \zeta_2, \dots$ , be a strictly stationary and ergodic sequence of random variables,  $\mathbf{P}\{\zeta_i = j\} = \mathbf{P}\{\zeta = j\}$ ,  $\zeta^{lower} \leq j \leq \zeta^{upper}$ . If  $\xi_i$  is the number of messages in the queue immediately before arrival of the  $i$ th message, then the message is lost if  $\xi_i > \zeta_i$ . Otherwise it joins the queue. We assume that  $\xi_1 = 0$ .

The existence of the stationary queue-length distribution, i.e.,

$$(1.1) \quad \mathbf{P}\{\bar{q} = j\} = \lim_{i \rightarrow \infty} \mathbf{P}\{\xi_i = j | \xi_1 < \infty\}, \quad j = 0, 1, \dots, \zeta^{upper},$$

is shown in the following. The special case when  $\mathbf{P}\{\nu_i = l\} = 1$  leads to the standard  $M/GI/1/n$  queueing system, where  $n = \lfloor N/l \rfloor$  is the integer part of  $N/l$ .

<sup>1</sup>For example, one of such technologies was developed in Orika Optical Networks Limited, where the author worked during 2000–2001.

It is also assumed that each message is marked with probability  $p$ . We study the asymptotic behavior of the loss probability under assumptions that  $\mathbf{E}\zeta$  increases to infinity and  $p$  vanishes. The details of these assumptions are clarified in the following consideration. The loss probability is the probability that the message is either marked or lost because of overflowing the queue. We study the cases where the traffic (offered load)  $\varrho = \lambda b$  is less than, equal to, and greater than 1.

**1.3. Advantages of the approach and methodology.** Our approach is based on the asymptotic analysis of the loss queueing systems in the earlier paper of the author (see Abramov [3]). The main method is an application of modern Tauberian theorems with remainder. For the relevant works devoted to asymptotic analysis of the loss and controlled systems with Poisson input, see Abramov [1], [2], Tomkó [18], and other papers. The asymptotic analysis of the  $GI/M/1/n$  queueing system was studied in [4], [9], [10]. The advantages of the approach of the present paper are the following.

*First*, our model is more general than the model from the aforementioned papers: This paper discusses the case of a non-Markovian buffer model where a message contains a random batch of packets, while the aforementioned papers studied a Markovian model with fixed batch size.

*Second*, the work in [6], [7], [11], [13] discusses a more difficult problem of consecutive losses, remaining in a framework of the standard  $M/M/1/n$  queueing system. The present paper flexibly discusses the stationary losses for a nonstandard queueing model with the random number of waiting places. That queueing system belongs to the special class of queueing systems with losses that is exactly defined below.

*Third*, our asymptotic analysis is much simpler than that of the other papers; our final results and their representation are simple and clear as well.

The traditional approach to asymptotic analysis, based on the final value theorem for  $z$  transform, enables us to obtain the main term of asymptotic relation and, in certain cases, a remainder. The modern Tauberian theorems enable us to obtain stronger asymptotic relations using some additional assumptions. These additional assumptions are realistic for the queueing systems considered here, and our asymptotic results are stronger than the earlier asymptotic results obtained for the  $M/GI/1/n$  queueing system with the aid of the final value theorem for  $z$  transform (see relation (4.15) for its comparison with (4.14)). For some other results related to asymptotic analysis of the  $M/GI/1/n$  and  $GI/M/1/n$  queueing systems with the aid of the final value theorem, see the bibliography notes and references in Abramov [1].

**1.4. What is the main result in this paper?** The paper contains a number of theoretical results on the asymptotic behavior of characteristics of the busy period of the system (section 4) and loss probability (section 5). These theoretical results are then used to conclude the effect of adding redundant packets in order to decrease the loss probability.

Although the theoretical results of the paper, related to the cases where the offered load  $\varrho < 1$ , are standard, the conclusion about adding redundancy is extremely simple and interesting nevertheless. Namely, the stationary loss probability is expressed only via the probability that there is a corrupted packet in the message. This enables us to conclude that adding a number of redundant packets can decrease the loss probability *with the rate of geometric progression* while  $\varrho < 1$ .

Then the case when  $\varrho$  is close to 1 is very important for the performance analysis. For example, it can be a result of adding a number of redundant packets when initially

$\varrho < 1$ . That is, to achieve a maximum decrease in the loss probability we allow an increase in the offered load up to the critical value.

Therefore, the results on redundancy, related to the case where  $\varrho = 1 + \varepsilon$  ( $\varepsilon > 0$ ) is slightly greater than 1, are extremely important. The usefulness of the case  $\varrho = 1 + \varepsilon$  is that it enables us to obtain more exact conclusions on redundancy based on asymptotic results with remainder. Then, the usefulness of the purely theoretical case  $\varrho > 1$  is that it is an intermediate result helping us to study the transient behavior, related to the case  $\varrho = 1 + \varepsilon$  for small  $\varepsilon > 0$ .

**1.5. Conclusion on adding redundant packets.** The results of the paper enable us to make conclusions on the effect of adding redundant packets as follows. Let  $\varrho$  denote the offered load of the system before adding a redundant packet, and let  $\check{\varrho} > \varrho$  be the value of the offered load after adding a redundant packet. While  $\check{\varrho}$  remains not greater than 1, adding redundant packets is profitable. It decreases the loss probability with the rate of a geometric progression. Adding a redundant packet remains profitable if the value  $\check{\varrho} = 1 + \varepsilon$ , where  $\varepsilon$  is a small value of a higher order than  $p$ . In some cases adding a redundant packet decreases the loss probability even when the value  $\varepsilon$  has the same order as  $p$ . These cases are studied in section 6.

**1.6. The organization of the paper.** The paper is organized as follows. There are six sections, with the first an introduction. In section 2 we introduce the class of queueing systems with a random number of waiting places and study the characteristics of the system busy period. The results on the expectations of random variables of the busy period (the number of processed messages, the number of refused messages, etc.) are given by Lemma 2.1. In section 3 we present a number of auxiliary results and the Tauberian theorems with remainder. These results are then used to prove a number of theorems on the asymptotic behavior of the characteristics of the system given on a busy period which in turn are given in section 4. Section 5 presents the results on asymptotic behavior of the loss probabilities under different assumptions. In section 6 we discuss adding redundancy. The central question here is, *How is the loss probability decreased or increased if we add redundant packets into the message?*

**2. Characteristics of the system given on a busy period.** The aim of this section is to deduce the explicit representations for characteristics of the system during a busy period such as the expected duration of a busy period, expected number of served and lost customers during a busy period, and so on. The queueing system described in section 1.2 is not standard, and the explicit representation for its characteristics cannot be obtained traditionally. Therefore, below we introduce a special class of queueing systems  $\Sigma$  containing the system studied in the paper and described in section 1.2. It will be shown in this section that the above characteristics are the same for all queueing systems of the class  $\Sigma$ . Hence, one can take any queueing system, a representative of class  $\Sigma$ , having a more simple structure than the original system, and study it instead of the original system.

For the sake of convenience, we denote by  $\mathcal{S}_1$  the system described in section 1.2. Let  $B(x)$  be the probability distribution function of a processing time (in the queueing terminology, a service time), and let  $\lambda$  be the parameter of Poisson input. We also set  $\varrho_j = \lambda^j \int_0^\infty x^j dB(x)$ ,  $j = 1, 2, \dots$ , and  $\varrho_1 = \varrho$ .

In order to study the characteristics of the system  $\mathcal{S}_1$  we introduce a set of systems  $\Sigma$  containing  $\mathcal{S}_1$  as an element. The set  $\Sigma$  is a set of  $M/GI/1$  queueing systems where  $\lambda$  is the rate of Poisson input,  $B(x)$  is the probability distribution function of a service time, and the family of sequences  $\{\zeta_i\}$  is more general than in  $\mathcal{S}_1$ . Each sequence  $\zeta_1$ ,

$\zeta_2, \dots$  is a family of identically distributed random variables, governing the rejection process and having the same distribution as the random variable  $\zeta$ . If this sequence is as defined in section 1.2, then we have a description of our system  $\mathcal{S}_1$ . In order to define the set  $\Sigma$  more exactly, we use the notation for the queueing system  $\mathcal{S}_1$  and also introduce the following.

Let  $\xi_i$  denote the number of messages in the system  $\mathcal{S}_1$  immediately before arrival of the  $i$ th message,  $\xi_1 = 0$ , and let  $s_i$  denote the number of service completions between the  $i$ th and  $i + 1$ st arrivals. It is clear that

$$(2.1) \quad \xi_{i+1} = \xi_i - s_i + \mathbf{I}\{\xi_i \leq \zeta_i\},$$

where the term  $\mathbf{I}\{\xi_i \leq \zeta_i\}$  indicates that the  $i$ th message is accepted, and obviously  $s_i$  is not greater than  $\xi_i + \mathbf{I}\{\xi_i \leq \zeta_i\}$ .

Consider a new queueing system  $\mathcal{S}$  as above with the Poisson input rate  $\lambda$  and the probability distribution function of a service  $B(x)$ , but with the sequence  $\tilde{\zeta}_1, \tilde{\zeta}_2, \dots$ . Here we assume that the sequence  $\{\tilde{\zeta}_i\}$  is an *arbitrary dependent* sequence of random variables consisting of identically distributed random variables as the random variable  $\zeta$ . Let  $\tilde{\xi}_i$  denote the number of messages immediately before arrival of the  $i$ th message ( $\tilde{\xi}_1 = 0$ ), and let  $\tilde{s}_i$  denote the number of service completions between the  $i$ th and  $i + 1$ st arrivals. Thus, we assume that the initial conditions of both queueing systems  $\mathcal{S}_1$  and  $\mathcal{S}$  are the same:  $\xi_1 = \tilde{\xi}_1$ .

Analogously to (2.1) we have

$$(2.2) \quad \tilde{\xi}_{i+1} = \tilde{\xi}_i - \tilde{s}_i + \mathbf{I}\{\tilde{\xi}_i \leq \tilde{\zeta}_i\}.$$

*Definition.* We say that the queueing system  $\mathcal{S}$  belongs to the set  $\Sigma$  of queueing systems if  $\mathbf{E}\tilde{\xi}_i = \mathbf{E}\xi_i$ ,  $\mathbf{E}\tilde{s}_i = \mathbf{E}s_i$ , and  $\mathbf{P}\{\tilde{\xi}_i \leq \tilde{\zeta}_i\} = \mathbf{P}\{\xi_i \leq \zeta_i\}$  for all  $i \geq 1$ .

Consider an example of queueing systems belonging to the set  $\Sigma$ , where the sequence  $\{\tilde{\zeta}_i\}$  is strictly stationary but not ergodic. The example is a queueing system with  $\tilde{\zeta}_1 = \tilde{\zeta}_2 = \dots$ , which we denote by  $\mathcal{S}_2$ . The example below is artificial rather than realistic, however, its main goal is to help us to show the existence of necessary stationary queue-length probabilities for the queueing system  $\mathcal{S}_1$  and to obtain the explicit representations for those probabilities as well.

For  $\mathcal{S}_2$  we find by induction for all  $i \geq 1$  that

$$(2.3) \quad \mathbf{E}\tilde{s}_i = \mathbf{E}s_i,$$

$$(2.4) \quad \mathbf{P}\{\tilde{\xi}_i \leq \tilde{\zeta}_i\} = \mathbf{P}\{\xi_i \leq \zeta_i\},$$

and

$$(2.5) \quad \mathbf{E}\tilde{\xi}_i - \mathbf{E}\tilde{s}_i + \mathbf{P}\{\tilde{\xi}_i \leq \tilde{\zeta}_i\} = \mathbf{E}\xi_i - \mathbf{E}s_i + \mathbf{P}\{\xi_i \leq \zeta_i\}.$$

Relations (2.3)–(2.5) show that the queueing system  $\mathcal{S}_2 \in \Sigma$ . It follows from the definition that if the stationary loss probability exists for at most one of the queueing systems  $\mathcal{S} \in \Sigma$ , then it exists for all queueing systems of  $\Sigma$  and it is the same. Then, the properties of the queueing system  $\mathcal{S}_2$  enable us to conclude similar properties of all queueing systems belonging to the set  $\Sigma$ , including  $\mathcal{S}_1$ . For example, it is not difficult to show that the expected busy period is the same for all queueing systems of the class  $\Sigma$ . Indeed, let  $\tilde{A}$ ,  $\tilde{S}$ , and  $\tilde{R}$  denote the number of arrived, served, and refused

customers (because of overflowing the buffer) during a busy cycle  $\tilde{C}$ , respectively. We have the equations

$$(2.6) \quad \mathbf{E}\tilde{A} = \mathbf{E}\tilde{S} + \mathbf{E}\tilde{R} = \lambda\mathbf{E}\tilde{C},$$

$$(2.7) \quad b\mathbf{E}\tilde{S} = \mathbf{E}\tilde{C} - \frac{1}{\lambda},$$

where  $b$  is the expected service time. Since the loss probability is the same for all queueing systems  $\mathcal{S} \in \Sigma$ , then the fraction  $\mathbf{E}\tilde{R}/\mathbf{E}\tilde{C}$  is the same for all  $\mathcal{S} \in \Sigma$  as well. Therefore, it follows from equations (2.6) and (2.7) that the expected duration of a busy period,  $\mathbf{E}\tilde{T} = \mathbf{E}\tilde{C} - \lambda^{-1}$ , is the same for all queueing systems  $\mathcal{S} \in \Sigma$ .

Recall that for queueing system  $\mathcal{S}_2$  we have  $\zeta_1 = \zeta_2 = \dots$ , i.e., the random variable  $\zeta$  is modeled once at the initial time moment. Let  $\tilde{T}_\zeta$  denote a busy period of this system. Then, the total expectation formula enables us to write

$$(2.8) \quad \mathbf{E}\tilde{T}_\zeta = \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{E}T_K \mathbf{P}\{\zeta = K\},$$

where  $\mathbf{E}T_K$  is the expected busy period of an  $M/GI/1/K$  queueing system with the same sequence of interarrival and service times, and  $\mathbf{P}\{\zeta = K\} = \mathbf{P}\{\zeta_j = K\}$ . In turn, the expectation  $\mathbf{E}T_K$  is determined from the following recurrence relation:

$$(2.9) \quad \mathbf{E}T_K = \sum_{j=0}^K \pi_j \mathbf{E}T_{K-j+1}, \quad \mathbf{E}T_0 = b, \quad \pi_i = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)$$

(see Tomkó [18], Cooper and Tilt [12], and Abramov [1], [3]), where  $b$  is the expectation of a service time.

Now, let  $T_\zeta$  denote a busy period for the queueing system  $\mathcal{S}_1$ . According to the above conclusion that  $\mathbf{E}T_\zeta = \mathbf{E}\tilde{T}_\zeta$ , and in view of (2.8), we have

$$(2.10) \quad \mathbf{E}T_\zeta = \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{E}T_K \mathbf{P}\{\zeta = K\},$$

where  $\mathbf{E}T_K$  are determined from (2.9).

Along with the notation  $T_\zeta$  for the busy period of the system  $\mathcal{S}_1$ , we let  $I_\zeta$  be an idle period and let  $P_\zeta$ ,  $M_\zeta$ ,  $R_\zeta$  be the characteristics of the system on a busy period: the number of processed messages, the number of marked messages, the number of refused messages, respectively. Here and later we use the following terminology. The term *refused* message is used for the case of overflowing the buffer. Then the term *lost* message is used for the case where a message is either refused or marked. The number of lost messages during a busy period is denoted by  $L_\zeta$ . Analogously, by loss probability we mean the probability when an arrival message is lost.

LEMMA 2.1. *For the expectations  $\mathbf{E}T_\zeta$ ,  $\mathbf{E}P_\zeta$ ,  $\mathbf{E}M_\zeta$ ,  $\mathbf{E}R_\zeta$  we have the following representations:*

$$(2.11) \quad \mathbf{E}P_\zeta = \frac{\lambda}{\varrho} \mathbf{E}T_\zeta,$$

$$(2.12) \quad \mathbf{E}M_\zeta = p\mathbf{E}P_\zeta,$$

$$(2.13) \quad \mathbf{E}R_\zeta = (\varrho - 1)\mathbf{E}P_\zeta + 1.$$

*Proof.* Relations (2.11) and (2.12) follow immediately from Wald’s identity. In order to prove (2.13) note that the number of arrivals during a busy cycle equals the number of processed messages during a busy period plus the number of refused messages during a busy period (see relation (2.6)). According to Wald’s identity the expected number of arrivals during a busy cycle equals  $\lambda(\mathbf{E}T_\zeta + \mathbf{E}I_\zeta)$ . Therefore taking into account that  $\mathbf{E}I_\zeta = \lambda^{-1}$  from (2.11), we have

$$\mathbf{E}R_\zeta = (\varrho - 1)\mathbf{E}P_\zeta + 1,$$

and the result is proved.  $\square$

For the alternative proof of (2.13) see Abramov [3]. (See also the proof in [5].)

**3. Auxiliary results. Tauberian theorems with remainder.** It is seen from relations (2.10) and (2.9) and Lemma 2.1 that the characteristics of the system during a busy period can be studied in a framework of the recurrence relation

$$(3.1) \quad Q_k = \sum_{i=0}^k r_i Q_{k-i+1},$$

where  $r_i$  are nonnegative numbers,  $r_0 + r_1 + \dots = 1$ ,  $r_0 > 0$ , and  $Q_0 \neq 0$  is an arbitrary real number. Below we recall a number of results on asymptotic behavior of that sequence (3.1).

The known results on representation (3.1) are asymptotic theorems by Takács [17]. Lemma 3.1 below joins two results by Takács: Theorem 5 of [17, p. 22] and relation (35) [17, p. 23]. The results of Takács [17] were then developed by Postnikov [14, sect. 25], [15, sect. 25] (see Lemma 3.2 and Lemma 3.3 below).

Let  $r(z) = \sum_{i=0}^\infty r_i z^i$ ,  $|z| \leq 1$ ,  $\gamma_m = r^{(m)}(1 - 0) = \lim_{z \uparrow 1} r^{(m)}(z)$  ( $r^{(m)}(z)$  is the  $m$ th derivative of  $r(z)$ ). Note that if we denote  $Q(z) = \sum_{i=0}^\infty Q_i z^i$ , then it follows from (3.1) that

$$Q(z) = \frac{Q_0 r(z)}{r(z) - z}.$$

LEMMA 3.1 (Takács [17]). *If  $\gamma_1 < 1$ , then*

$$(3.2) \quad \lim_{k \rightarrow \infty} Q_k = \frac{Q_0}{1 - \gamma_1}.$$

*If  $\gamma_1 = 1$  and  $\gamma_2 < \infty$ , then*

$$(3.3) \quad \lim_{k \rightarrow \infty} \frac{Q_k}{k} = \frac{2Q_0}{\gamma_2}.$$

*If  $\gamma_1 > 1$ , then*

$$(3.4) \quad \lim_{k \rightarrow \infty} \left( Q_k - \frac{Q_0}{\delta^k [1 - r'(\delta)]} \right) = \frac{Q_0}{1 - \gamma_1},$$

where  $\delta$  is the least (absolute) root of the equation  $z = r(z)$ .

LEMMA 3.2 (Postnikov [14], [15]). *Let  $\gamma_1 = 1$ ,  $\gamma_3 < \infty$ . Then as  $k \rightarrow \infty$ ,*

$$(3.5) \quad Q_k = \frac{2Q_0}{\gamma_2} k + O(\log k).$$

LEMMA 3.3 (Postnikov [14], [15]). *Let  $\gamma_1 = 1$ ,  $\gamma_2 < \infty$  and  $r_0 + r_1 < 1$ . Then as  $k \rightarrow \infty$ ,*

$$(3.6) \quad Q_{k+1} - Q_k = \frac{2Q_0}{\gamma_2} + o(1).$$

**4. Asymptotic results for characteristics of the system during a busy period.** This section provides a number of results on asymptotic behavior of characteristics of the system. The first three theorems are related to the case as  $N$  increases to infinity, where the cases  $\varrho < 1$ ,  $\varrho = 1$ , and  $\varrho > 1$  are considered. The next two theorems discuss the case when the value  $\varrho$  is close to the critical value 1, and as  $N \rightarrow \infty$ , it tends to 1. The last theorem of this section, Theorem 4.6, provides the asymptotic result for the special case when the number of packets in a message is a constant value.

Let us now study the asymptotic behavior of the expectations  $\mathbf{E}P_\zeta$ ,  $\mathbf{E}M_\zeta$ , and  $\mathbf{E}R_\zeta$ . We write  $\zeta = \zeta(N)$ , pointing out the dependence on parameter  $N$ . As the buffer size  $N$  increases to infinity, both  $\zeta^{lower}$  and  $\zeta^{upper}$  tend to infinity, and together with them,  $\zeta(N)$  a.s. tends to infinity. Then we have the following.

THEOREM 4.1. *If  $\varrho < 1$ , then*

$$(4.1) \quad \lim_{N \rightarrow \infty} \mathbf{E}P_{\zeta(N)} = \frac{1}{1 - \varrho}.$$

*If  $\varrho = 1$  and  $\varrho_2 < \infty$ , then*

$$(4.2) \quad \lim_{N \rightarrow \infty} \frac{\mathbf{E}P_{\zeta(N)}}{\mathbf{E}\zeta(N)} = \frac{2}{\varrho_2}.$$

*If  $\varrho > 1$ , then*

$$(4.3) \quad \lim_{N \rightarrow \infty} \left[ \mathbf{E}P_{\zeta(N)} - \frac{1}{\mathbf{E}\varphi^{\zeta(N)} [1 + \lambda\beta'(\lambda - \lambda\varphi)]} \right] = \frac{1}{1 - \varrho},$$

where  $\beta(z) = \int_0^\infty e^{-zx} dB(x)$  and  $\varphi$  is the least (absolute) root of functional equation  $z - \beta(\lambda - \lambda z) = 0$ .

*Proof.* From (2.9), (2.10), and (2.11) we have

$$\mathbf{E}P_\zeta = \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{E}P_K \mathbf{P}\{\zeta = K\},$$

where

$$\mathbf{E}P_K = \sum_{j=0}^K \pi_j \mathbf{E}P_{K-j+1}, \quad \mathbf{E}P_0 = 1,$$

$$\pi_j = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x).$$

Then applying Lemma 3.1 we have the following. In the case  $\varrho < 1$ , taking into account that  $\zeta(N) \xrightarrow{a.s.} \infty$  as  $N \rightarrow \infty$ , we obtain

$$\lim_{N \rightarrow \infty} \mathbf{E}P_{\zeta(N)} = \lim_{N \rightarrow \infty} \mathbf{E}P_N = \frac{1}{1 - \varrho}.$$



Relation (4.1) is proved.

In the case  $\varrho_2 < \infty$  and  $\varrho = 1$  we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\mathbf{E}P_{\zeta(N)}}{N} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{P}\{\zeta(N) = K\} \mathbf{E}P_K \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{K=\zeta^{lower}}^{\zeta^{upper}} K \mathbf{P}\{\zeta(N) = K\} \frac{2}{\varrho_2} = \frac{2}{\varrho_2} \lim_{N \rightarrow \infty} \frac{\mathbf{E}\zeta(N)}{N}. \end{aligned}$$

Therefore,

$$\lim_{N \rightarrow \infty} \frac{\mathbf{E}P_{\zeta(N)}}{\mathbf{E}\zeta(N)} = \frac{2}{\varrho_2},$$

and relation (4.2) is proved.

In the case  $\varrho > 1$  for large  $N$  we obtain

$$\begin{aligned} \mathbf{E}P_{\zeta(N)} &= \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{P}\{\zeta(N) = K\} \mathbf{E}P_K \\ &= \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{P}\{\zeta(N) = K\} \frac{1}{\varphi^K [1 + \lambda\beta'(\lambda - \lambda\varphi)]} + \frac{1}{1 - \varrho} + o(1) \\ &= \frac{1}{\mathbf{E}\varphi^{\zeta(N)} [1 + \lambda\beta'(\lambda - \lambda\varphi)]} + \frac{1}{1 - \varrho} + o(1). \end{aligned}$$

Therefore,

$$\lim_{N \rightarrow \infty} \left[ \mathbf{E}P_{\zeta(N)} - \frac{1}{\mathbf{E}\varphi^{\zeta(N)} [1 + \lambda\beta'(\lambda - \lambda\varphi)]} \right] = \frac{1}{1 - \varrho},$$

and relation (4.3) is proved. Theorem 4.1 is completely proved.  $\square$

**THEOREM 4.2.** *If  $\varrho = 1$  and  $\varrho_3 < \infty$ , then*

$$(4.4) \quad \mathbf{E}P_{\zeta(N)} = \frac{2}{\varrho_2} \mathbf{E}\zeta(N) + O(\log N).$$

*Proof.* Applying Lemma 3.2, for large  $N$  we have

$$\begin{aligned} \mathbf{E}P_{\zeta(N)} &= \sum_{K=\zeta^{lower}}^{\zeta^{upper}} \mathbf{P}\{\zeta(N) = K\} \mathbf{E}P_K \\ &= \sum_{K=\zeta^{lower}}^{\zeta^{upper}} K \mathbf{P}\{\zeta(N) = K\} \frac{2}{\varrho_2} + O\{\mathbf{E}[\log \zeta(N)]\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\varrho_2} \mathbf{E}\zeta(N) + O\{\mathbf{E}[\log \zeta(N)]\} \\
 &= \frac{2}{\varrho_2} \mathbf{E}\zeta(N) + O(\log N).
 \end{aligned}$$

and we obtain relation (4.4). Theorem 4.2 is proved.  $\square$

In turn for  $\mathbf{E}R_\zeta$  we have the following theorem.

**THEOREM 4.3.** *If  $\varrho < 1$ , then*

$$(4.5) \quad \lim_{N \rightarrow \infty} \mathbf{E}R_{\zeta(N)} = 0.$$

*If  $\varrho = 1$ , then for all  $N \geq 0$*

$$(4.6) \quad \mathbf{E}R_{\zeta(N)} = 1.$$

*If  $\varrho > 1$ , then*

$$(4.7) \quad \lim_{N \rightarrow \infty} \left[ \mathbf{E}R_{\zeta(N)} - \frac{\varrho - 1}{\mathbf{E}\varphi^{\zeta(N)} [1 + \lambda\beta'(\lambda - \lambda\varphi)]} \right] = 0.$$

*Proof.* The proof of this theorem is analogous to that of the proof of Theorem 4.1. It follows by application of Lemma 3.1 and relation (2.13) of Lemma 2.1.  $\square$

**THEOREM 4.4.** *Let  $\varrho = 1 + \varepsilon$ ,  $\varepsilon > 0$ , and  $\varepsilon\zeta(N) \rightarrow C > 0$  a.s. as  $\varepsilon \rightarrow 0$  and  $N \rightarrow \infty$ . Assume also that  $\varrho_3 = \varrho_3(N)$  is a bounded sequence, and there exists  $\tilde{\varrho}_2 = \lim_{N \rightarrow \infty} \varrho_2(N)$ . Then*

$$(4.8) \quad \mathbf{E}P_{\zeta(N)} = \frac{e^{2C/\tilde{\varrho}_2} - 1}{\varepsilon} + O(1),$$

$$(4.9) \quad \mathbf{E}R_{\zeta(N)} = e^{2C/\tilde{\varrho}_2} + o(1).$$

*Proof.* It was shown in Subhankulov [16, p. 326], that if  $\varrho = 1 + \varepsilon$ ,  $\varepsilon > 0$ ,  $\varepsilon \rightarrow 0$ ,  $\varrho_3(N)$  is a bounded sequence, and there exists  $\tilde{\varrho}_2 = \lim_{N \rightarrow \infty} \varrho_2(N)$ , then

$$(4.10) \quad \varphi = 1 - \frac{2\varepsilon}{\tilde{\varrho}_2} + O(\varepsilon^2).$$

Applying (4.10) after some algebra we have

$$(4.11) \quad 1 + \lambda\beta'(\lambda - \lambda\varphi) = \varepsilon + O(\varepsilon^2).$$

Then the statements of the theorem follow by applying expansions (4.10) and (4.11) to (4.3) and (4.7).  $\square$

**THEOREM 4.5.** *Let  $\varrho = 1 + \varepsilon$ ,  $\varepsilon > 0$ , and  $\varepsilon\zeta(N) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $N \rightarrow \infty$ . Assume also that  $\varrho_3 = \varrho_3(N)$  is a bounded sequence, and there exists  $\tilde{\varrho}_2 = \lim_{N \rightarrow \infty} \varrho_2(N)$ . Then*

$$(4.12) \quad \mathbf{E}P_{\zeta(N)} = \frac{2}{\tilde{\varrho}_2} \mathbf{E}\zeta(N) + O(1),$$

$$(4.13) \quad \mathbf{E}R_{\zeta(N)} = 1 + o(1).$$

*Proof.* The results follow by expanding (4.8) and (4.9) for small  $C$ .  $\square$

*Special case.* If each message contains the same number of packets, say  $l$ , then we have the usual  $M/GI/1/n$  queueing system, where  $n = \lfloor N/l \rfloor$  is the integer part of  $N/l$ . For that queueing system all the results in Theorems 4.1–4.5 hold, by replacing  $\zeta(N)$  (or  $\mathbf{E}\zeta(N)$ ) by  $n$ .

For example, asymptotic relation (4.7) appears as

$$(4.14) \quad \lim_{n \rightarrow \infty} \left( \mathbf{E}R_n - \frac{\varrho - 1}{\varphi^n [1 + \lambda\beta'(\lambda - \lambda\varphi)]} \right) = 0.$$

Notice that using the final value theorem for  $z$  transform, Azlarov and Tahirov [8] obtain the estimation

$$(4.15) \quad \mathbf{E}R_n = \frac{\varrho - 1}{\varphi^n [1 + \lambda\beta'(\lambda - \lambda\varphi)]} \left[ 1 + O\left(\frac{2\varphi}{1 + \varphi}\right)^n \right],$$

weaker than (4.14).

The theorem below is related to the case of the usual queueing systems only, when the number of packets in a message is fixed. Namely, we have the following.

**THEOREM 4.6.** *If  $\varrho = 1$  and  $\varrho_2 < \infty$ , then*

$$(4.16) \quad \mathbf{E}P_{n+1} - \mathbf{E}P_n = \frac{2}{\varrho_2} + o(1), \quad n \rightarrow \infty,$$

where the index  $n + 1$  says that  $P_{n+1}$  is the number of processed messages during a busy period of the  $M/GI/1/n + 1$  queueing system.

*Proof.* The result will follow from Lemma 3.3 if we show that  $\beta(\lambda) - \lambda\beta'(\lambda) < 1$ . Taking into account that for each  $\lambda > 0$ ,

$$(4.17) \quad \begin{aligned} \sum_{i=0}^{\infty} \frac{(-\lambda)^i}{i!} \beta^{(i)}(\lambda) &= \sum_{i=0}^{\infty} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x) \\ &= \int_0^{\infty} \sum_{i=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x) = 1, \end{aligned}$$

and all terms

$$\pi_i = \frac{(-\lambda)^i}{i!} \beta^{(i)}(\lambda)$$

are nonnegative, from (4.17) we find that

$$(4.18) \quad \beta(\lambda) - \lambda\beta'(\lambda) \leq 1.$$

Thus, the required statement will be proved if we show that for some  $\lambda_0 > 0$  the equality

$$(4.19) \quad \beta(\lambda_0) - \lambda_0\beta'(\lambda_0) = 1$$

is not a case. Indeed, since the function  $\beta(\lambda) - \lambda\beta'(\lambda)$  is an analytic function, then according to the maximum absolute value principle for analytic functions,  $\beta(\lambda) - \lambda\beta'(\lambda) = 1$  holds for all  $\lambda > 0$ . Therefore identity (4.19) means that  $\pi_i = 0$  for all  $i \geq 2$  and for all  $\lambda > 0$ . Therefore, (4.19) is valid if and only if  $\beta(\lambda)$  is a linear function, i.e.,  $\beta(\lambda) = c_0 + c_1\lambda$ ,  $c_0$  and  $c_1$  are some constants. However, since  $|\beta(\lambda)| \leq 1$  we obtain  $c_0 = 1$  and  $c_1 = 0$ , and  $\beta(\lambda) \equiv 1$ . This is the trivial case where the probability distribution function  $B(x)$  is concentrated in point 0. Therefore (4.19) is not a case, and  $\beta(\lambda) - \lambda\beta'(\lambda) < 1$ . The theorem is proved.  $\square$

**5. Asymptotic theorems for the loss probabilities.** In this section we study the asymptotic behavior of the loss probability by using renewal arguments. The results of this section correspond to those of the previous section. We discuss the behavior of the system for the same cases as  $N \rightarrow \infty$ , as well as when the parameter  $\rho$  is close to the critical value 1 and tends to 1 as  $N \rightarrow \infty$ . The theorems of this section are important for our conclusion on adding redundancy, which is given in the next section.

According to renewal arguments the loss probability is determined as

$$(5.1) \quad \Pi_{\zeta} = \frac{\mathbf{E}L_{\zeta}}{\mathbf{E}R_{\zeta} + \mathbf{E}P_{\zeta}} = \frac{\mathbf{E}R_{\zeta} + \mathbf{E}M_{\zeta}}{\mathbf{E}R_{\zeta} + \mathbf{E}P_{\zeta}} = \frac{\mathbf{E}R_{\zeta} + p\mathbf{E}P_{\zeta}}{\mathbf{E}R_{\zeta} + \mathbf{E}P_{\zeta}}.$$

(Recall that  $L_{\zeta}$  is the number of lost messages during a busy period.)

THEOREM 5.1. *If  $\rho < 1$*

$$(5.2) \quad \lim_{N \rightarrow \infty} \Pi_{\zeta(N)} = p.$$

(Recall that  $p$  is the probability that a message is erroneous because one of its packets is corrupted.)

Limiting relation 5.2 is also valid when  $\rho = 1$  and  $\rho_2 < \infty$ .

If  $\rho > 1$ , then

$$(5.3) \quad \Pi_{\zeta(N)} = \frac{p + \rho - 1}{\rho} \frac{(\rho - 1) + p[1 + \lambda\beta'(\lambda - \lambda\varphi)]\mathbf{E}\varphi^{\zeta(N)}}{(\rho - 1) + [1 + \lambda\beta'(\lambda - \lambda\varphi)]\mathbf{E}\varphi^{\zeta(N)}} + o(\mathbf{E}\varphi^{\zeta(N)}).$$

*Proof.* The proof follows from Theorems 4.1 and 4.3.  $\square$

THEOREM 5.2. *If  $\rho = 1$  and  $\rho_3 < \infty$ , then as  $N \rightarrow \infty$*

$$(5.4) \quad \Pi_{\zeta(N)} = p + \frac{(1-p)\rho_2}{2\mathbf{E}\zeta(N)} + O\left(\frac{\log N}{N^2}\right).$$

*Proof.* From (5.1) we have

$$(5.5) \quad \begin{aligned} \Pi_{\zeta(N)} &= \frac{\mathbf{E}R_{\zeta(N)}}{\mathbf{E}R_{\zeta(N)} + \mathbf{E}P_{\zeta(N)}} + \frac{p\mathbf{E}P_{\zeta(N)}}{\mathbf{E}R_{\zeta(N)} + \mathbf{E}P_{\zeta(N)}} \\ &= \frac{1}{1 + \mathbf{E}P_{\zeta(N)}} + \frac{p\mathbf{E}P_{\zeta(N)}}{1 + \mathbf{E}P_{\zeta(N)}}. \end{aligned}$$

As  $N \rightarrow \infty$  from Theorem 4.2 we obtain

$$(5.6) \quad \frac{1}{1 + \mathbf{E}P_{\zeta(N)}} = \frac{\rho_2}{2\mathbf{E}\zeta(N)} + O\left(\frac{\log N}{N^2}\right),$$

$$(5.7) \quad \frac{p\mathbf{E}P_{\zeta(N)}}{1 + \mathbf{E}P_{\zeta(N)}} = p - \frac{p\rho_2}{2\mathbf{E}\zeta(N)} + O\left(\frac{\log N}{N^2}\right).$$

Combining these two asymptotic relations (5.6) and (5.7) we obtain the statement of Theorem 5.2. Theorem 5.2 is proved.  $\square$

*Note.* Under assumptions of Theorem 5.2 assume additionally that  $p \rightarrow 0$ . If  $pN \rightarrow C > 0$ , then

$$\Pi_{\zeta(N)} = \frac{C}{N} + \frac{\varrho_2}{2\mathbf{E}\zeta(N)} + O\left(\frac{\log N}{N^2}\right).$$

If  $pN \rightarrow 0$ , then

$$\Pi_{\zeta(N)} = \frac{\varrho_2}{2\mathbf{E}\zeta(N)} + O\left(p + \frac{\log N}{N^2}\right).$$

The theorem below also assumes that  $p \rightarrow 0$ . Our result here is the following.

**THEOREM 5.3.** *Let  $\varrho = 1 + \varepsilon$ ,  $\varepsilon > 0$ , and  $\varepsilon\zeta(N) \rightarrow C > 0$  as  $\varepsilon \rightarrow 0$  and  $N \rightarrow \infty$ , and  $p \rightarrow 0$ . Assume also that  $\varrho_3 = \varrho_3(N)$  is a bounded sequence, and there exists  $\tilde{\varrho}_2 = \lim_{n \rightarrow \infty} \varrho_2(N)$ .*

(i) *If  $p/\varepsilon \rightarrow D \geq 0$ , then we have*

$$(5.8) \quad \Pi_{\zeta(N)} = \left( D + \frac{e^{2C/\tilde{\varrho}_2}}{e^{2C/\tilde{\varrho}_2} - 1} \right) \varepsilon + o(\varepsilon).$$

(ii) *If  $p/\varepsilon \rightarrow \infty$ , then we have*

$$(5.9) \quad \Pi_{\zeta(N)} = p + O(\varepsilon).$$

*Proof.* In the case (i) we have

$$(5.10) \quad p\mathbf{E}P_{\zeta(N)} + \mathbf{E}R_{\zeta(N)} = (D + 1)e^{2C/\tilde{\varrho}_2} - D + o(1),$$

and

$$(5.11) \quad \mathbf{E}P_{\zeta(N)} + \mathbf{E}R_{\zeta(N)} = \frac{e^{2C/\tilde{\varrho}_2} - 1}{\varepsilon} + O(1).$$

Therefore from (5.10) and (5.11) we have

$$\Pi_{\zeta(N)} = \left( D + \frac{e^{2C/\tilde{\varrho}_2}}{e^{2C/\tilde{\varrho}_2} - 1} \right) \varepsilon + o(\varepsilon),$$

and relation (5.8) is proved.

In the case (ii) we have

$$(5.12) \quad p\mathbf{E}P_{\zeta(N)} + \mathbf{E}R_{\zeta(N)} = \frac{pC}{\varepsilon} + O(1),$$

and

$$(5.13) \quad \mathbf{E}P_{\zeta(N)} + \mathbf{E}R_{\zeta(N)} = \frac{c}{\varepsilon} + O(1),$$

where  $c = \exp(2C/\tilde{\varrho}_2)/(\exp(2C/\tilde{\varrho}_2) - 1)$ . Relation (5.9) follows.  $\square$

**THEOREM 5.4.** *Let  $\varrho = 1 + \varepsilon$ ,  $\varepsilon > 0$ , and  $\varepsilon\zeta(N) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $N \rightarrow \infty$ , and  $p \rightarrow 0$ . Assume also that  $\varrho_3 = \varrho_3(N)$  is a bounded sequence, and there exists  $\tilde{\varrho}_2 = \lim_{n \rightarrow \infty} \varrho_2(N)$ .*

(i) If  $p/\varepsilon \rightarrow D \geq 0$ , then we have

$$(5.14) \quad \Pi_{\zeta(N)} = p + \frac{\tilde{\varrho}_2}{2\mathbf{E}\zeta(N)} + o\left(\frac{1}{N}\right).$$

(ii) If  $p/\varepsilon \rightarrow \infty$ , then we have (5.9).

*Proof.* The proof of (5.14) follows by expanding (5.8) for small  $C$ . The proof in case (ii) trivially follows from (5.12) and (5.13).  $\square$

*Special case.* In the case where each message contains exactly  $l$  packets,  $n = \lfloor N/l \rfloor$ , we obtain the following:

THEOREM 5.5. *If  $\varrho = 1$  and  $\varrho_2 < \infty$ , then as  $n \rightarrow \infty$*

$$(5.15) \quad \Pi_{n+1} - \Pi_n = \frac{\frac{1}{n(n+1)} \frac{2}{\varrho_2} (p-1)}{\left(\frac{2}{\varrho_2} + \frac{1}{n+1}\right)\left(\frac{2}{\varrho_2} + \frac{1}{n}\right)} + o\left(\frac{1}{n^2}\right).$$

*Proof.* The proof follows by applying Theorem 4.5 and taking into account the fact that  $\mathbf{E}R_n = 1$  for all  $n \geq 0$  (see [3] or Lemma 2.1).  $\square$

**6. Adding redundant packets.** We now investigate the effect of adding redundant packets. We assume that adding a redundant packet to the message decreases the probability  $p$  that a message is corrupted and increases the offered load and the number of packets in a message. The new parameters of the system after adding a redundant packet are denoted by adding the symbol  $\checkmark$  above. For example,  $\check{p}$  is a probability that a message contains a corrupted packet and  $\check{\varrho}$  is the offered load. It follows from Theorem 5.1 that if  $\check{\varrho} \leq 1$  the stationary loss probability coincides with  $\check{p}$ . This means that if adding a redundant packet to the message decreases the probability  $p$  by  $\gamma$  times, then the same effect is achieved with the loss probability. Thus, adding a number of redundant packets while  $\varrho < 1$  can decrease the loss probability geometrically.

In the case where both  $\varrho > 1$  and  $\check{\varrho} > 1$ , adding a redundant packet to the message changes the stationary loss probability to approximately

$$\frac{\varrho(\check{p} + \check{\varrho} - 1)}{\check{\varrho}(p + \varrho - 1)}.$$

In practice the values  $p$  and  $\check{p}$  are small, and even if adding redundant packets can slightly decrease the stationary loss probability, the effect of that action is not considerable.

The case where  $\varrho < 1$  and  $\check{\varrho} > 1$  is especially interesting if  $\check{\varrho} = 1 + \delta$ , and  $\delta$  is a small value. For example, if  $\delta$  is so small that both  $\delta\zeta(N)$  and  $\delta/p$  are also negligible, then a redundant packet decreases the loss probability by approximately the same amount as in the case when both  $\varrho < 1$  and  $\check{\varrho} < 1$ . However, if  $\delta$  is of the same order as  $p$  or  $1/\zeta(N)$ , then the special analysis based on the corresponding cases of Theorems 5.3 and 5.4 is necessary. Here we do not provide the details.

Let us consider the cases when both  $\varrho > 1$  and  $\check{\varrho} > 1$ , where  $\varrho = 1 + \varepsilon$  and  $\check{\varrho} = 1 + \check{\varepsilon}$ , and  $\varepsilon$  and  $\check{\varepsilon}$  are small values as in Theorem 5.3, both satisfying (i). Then the stationary loss probability is changed to approximately

$$(6.1) \quad \frac{e^{2C/\tilde{\varrho}_2} - 1}{e^{2\check{C}/\tilde{\varrho}_2} - 1} \frac{(e^{2\check{C}/\tilde{\varrho}_2} - 1)\check{p} + e^{2\check{C}/\tilde{\varrho}_2}\check{\varepsilon}}{(e^{2C/\tilde{\varrho}_2} - 1)p + e^{2C/\tilde{\varrho}_2}\varepsilon}$$

times.

For the sake of simplicity let us assume that  $\check{C}/\check{\varrho}_2 = C/\tilde{\varrho}_2$ . Then (6.1) reduces to

$$(6.2) \quad \frac{(e^{2C/\tilde{\varrho}_2} - 1)\check{p} + e^{2C/\tilde{\varrho}_2}\check{\varepsilon}}{(e^{2C/\tilde{\varrho}_2} - 1)p + e^{2C/\tilde{\varrho}_2}\varepsilon}.$$

If we assume that

$$p - \check{p} = \frac{e^{2C/\tilde{\varrho}_2}}{e^{2C/\tilde{\varrho}_2} - 1}(\check{\varepsilon} - \varepsilon),$$

then the stationary loss probability remains at approximately the same value, and if

$$p - \check{p} > \frac{e^{2C/\tilde{\varrho}_2}}{e^{2C/\tilde{\varrho}_2} - 1}(\check{\varepsilon} - \varepsilon),$$

then the stationary loss probability decreases, otherwise if

$$p - \check{p} < \frac{e^{2C/\tilde{\varrho}_2}}{e^{2C/\tilde{\varrho}_2} - 1}(\check{\varepsilon} - \varepsilon),$$

then the stationary loss probability increases.

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